PHYS 320 ANALYTICAL MECHANICS

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Fourier series:

- Full differential equation of motion:

$$m\ddot{x}(t) + c\dot{x}(t) + k\,x(t) = F_{ext}(t)$$

- forcing function written as Fourier series:

$$F_{ext}(t) = \sum_{n=1}^{\infty} F_n = \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

- solution to full diff. eqn. is linear superposition of solutions to the n^{th} differential equation:

$$m\ddot{x}_n(t) + c\dot{x}_n(t) + k\,x_n(t) = F_n(t)$$

$$x(t) = \sum_{n=1}^{\infty} x_n(t) = \sum_{n=1}^{\infty} A_n \cos(n\omega t - \delta_n)$$

where $A_n = \dfrac{b_n / m}{\left[(\omega_o^2 - n^2\omega^2)^2 + 4\gamma^2\, n^2\omega^2\right]^{1/2}}$ and $\delta_n = \tan^{-1}\left(\dfrac{2\gamma n\omega}{\omega_o^2 - n^2\omega^2}\right)$

Fourier Series

For a function defined on the interval [-L, L], where

$$f(x') = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n \pi x'}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n \pi x'}{L}\right).$$

where

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x')\, dx'$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x') \cos\left(\frac{n \pi x'}{L}\right) dx'$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x') \sin\left(\frac{n \pi x'}{L}\right) dx'.$$

Similarly, the function is instead defined on the interval $[0, 2L]$, the above equations simply become

$$a_0 = \frac{1}{L} \int_{0}^{2L} f(x')\, dx'$$

$$a_n = \frac{1}{L} \int_{0}^{2L} f(x') \cos\left(\frac{n \pi x'}{L}\right) dx'$$

$$b_n = \frac{1}{L} \int_{0}^{2L} f(x') \sin\left(\frac{n \pi x'}{L}\right) dx'.$$

Fourier Series

For our context, rewrite in terms of period, T:

$$f(t) = \frac{1}{2}a_o + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt, \quad n = 0,1,2,\ldots$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)dt, \quad n = 1,2,\ldots$$

$$T = \frac{2\pi}{\omega}$$

Fourier series: sawtooth driving force

- Solutions are $x(t) = \sum_{n=1}^{\infty} x_n(t) = \sum_{n=1}^{\infty} A_n \cos(n\omega t - \delta_n)$

where $A_n = \dfrac{b_n / m}{\left[(\omega_o^2 - n^2\omega^2)^2 + 4\gamma^2 n^2\omega^2\right]^{1/2}}$, $\delta_n = \tan^{-1}\left(\dfrac{2\gamma n\omega}{\omega_o^2 - n^2\omega^2}\right)$

$$b_n = \frac{A}{n\pi}(-1)^{n+1}$$

$$b_1 = \frac{A}{\pi}, \quad b_2 = \frac{-A}{2\pi}, \quad b_3 = \frac{A}{3\pi}, \quad etc.$$

So, we have

$$A_1 = \frac{A/\pi m}{\left[(\omega_o^2 - \omega^2)^2 + 4\gamma^2\omega^2\right]^{1/2}}, \quad \delta_1 = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_o^2 - \omega^2}\right)$$

$$A_2 = \frac{-A/2\pi m}{\left[(\omega_o^2 - 4\omega^2)^2 + 4\gamma^2(4\omega^2)\right]^{1/2}}, \quad \delta_2 = \tan^{-1}\left(\frac{2\gamma(2\omega)}{\omega_o^2 - 4\omega^2}\right)$$

$$A_3 = \frac{-A/2\pi m}{\left[(\omega_o^2 - 9\omega^2)^2 + 4\gamma^2(9\omega^2)\right]^{1/2}}, \quad \delta_3 = \tan^{-1}\left(\frac{2\gamma(3\omega)}{\omega_o^2 - 9\omega^2}\right)$$

et cetera!

if any are near ω_r then get strong response!